International Journal of Theoretical Physics, Vol. 45, No. 11, November 2006 (© 2006)
DOI: 10.1007/s10773-006-9177-1

Geometric Quantum Computing and Dissipation Models

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Received January 19, 2006; Accepted March 13, 2006
Published Online: May 17, 2006

Based on a simple periodical external field model, we investigate the impact of standard Leggett's dissipation on the Berry's phase, which is necessary for any practical implementation of geometric phase gate. It is found that the environmental noise, including the thermal and vacuum parts, could lead to a decaying term in the matrix of Berry's phase, which corresponds to the decoherence process of a qubit as a function of both time and temperature. A new type of two-level-system reservoir is also discussed, it is shown that the decaying term only depends on time, but not on temperature. A concrete case is exhibited by using the 1D Ohmic function.

KEY WORDS: geometric phase; dissipation models.

1. INTRODUCTION

Currently, there has been an explosive growth of interests in quantum computers and universal quantum gates because of its novel ability in processing information (Ekert and Jozsa, 1996; Aharonov, 1998; Steane, 1998; Bennett and DiVincenzo, 2000). Many physical systems have been used to implement various quantum gates, in which the NMR technique seems to be a powerful one in the present laboratory (Jones, 2000). Most recently, Jones *et al.* (2000) and Ekert *et al.* (2000) presented a novel scheme to realize the phase gate and then quantum computing was purely performed by the geometric Berry's phase. The dynamic phase was diminished by using the spin-echo technique in NMR experiments and the geometric phase was proved to be resilient for the errors in the amplitude of the external RF field. By using a new designed sequence of simple operations

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with an additional vertical magnetic field, the conditional geometric phase shift was also generalized to the non-adiabatic case (Wang and Matsumoto, 2001a,b). However, an attractive problem about the effects of environmental noise on the acquired geometric phase in the physical system has not been investigated and its robust in the presence of dissipation still deserves some theoretical analysis for any possible applications of their scheme to the practical quantum computing.

Historically, the investigations about the influence of dissipation on quantum coherence were focused on the spin-boson model, which is largely due to the pioneering work of Leggett *et al.* (1987; for an analysis of the drawback of this standard model, see, e.g., Vacchini, 2000). Through a different formulation of environment, Shao *et al.* presented a new type of two-level-system reservoir which, in the weak coupling, corresponds to the realistic physical situation of a spin interacting with its surrounding (effectively) independent spin modes (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000). The effect that temperature helps the system suppress decoherence was revealed (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000), as they suspected, may favor the quantum computing efforts.

In this paper, by using the relatively simple versions of these two dissipation models respectively, we hope to gain some insights on the geometric phase in the presence of dissipation. Our method is based on the concept of Berry's phase matrix. After showing the entanglement of a qubit with its environment for static case within the standard Leggett's model (1987) and Vacchini (2000), we firstly study the impact of Leggett's dissipation on the acquired geometric phase of the spin trapped in a periodical magnetic field. It is shown that the environmental noise, including the thermal and vacuum parts, leads to a decaying term in the matrix of Berry's phase, which corresponds to the decoherence process of a qubit as a function of both time and of temperature. Then we investigate the similar problem by using the new type of two-level-system reservoir, which leads to a decaying term exhibiting a different temperature-independent behavior.

2. BERRY'S PHASE MATRIX

Before considering the possible dissipation effects in the concerned system, we would like to introduce the simple generalization of Berry's phase from the evolution of wave function to the evolution of the reduced density matrix. It is well known that, if the system studied is not coupled to other systems, and if it starts from a mixed state, then the properties of the system is totally described by the density matrix $\rho(t)$ that complies with Liouville equation. Suppose that the Hamiltonian is $\hat{H}(t) = \hat{H}(R(t))$, where $R(t)$ is some periodic parameter $R(t + \mathcal{T}) = R(t)$ and $\hat{H}(R)|\psi_n(R)\rangle = E_n(R)|\psi_n\rangle$ is the so-called instantaneously stationary Schrödinger equation. Without losing generality, we set

$\rho(0) = \sum_{m,n}^{N} c_{mn}|\psi_m(0)\rangle\langle\psi_n(0)|$. The density matrix at time t reads

$$\rho(t) = \sum_{m,n}^{N} c_{mn}|\phi_m(t)\rangle\langle\phi_n(t)|, \tag{1}$$

where $\phi_m(t) = e^{-i\int_0^t dt' E_m(t')} + e^{i\zeta_m(t)}|\psi_m(R(t))\rangle$ and $\zeta_m(t)$ being the ordinary Berry's phase. In principle, the Berry's phase matrix $\gamma(t)$ could be defined from

$$\rho(t) \equiv e^{i\gamma(t)}\rho_{\text{ad}}(t), \tag{2}$$

where

$$\rho_{\text{ad}}(t) = \sum_{m,n}^{N} c_{mn}e^{-i\int_0^t dt'[(E_m(t')-E_n(t')]}|\psi_m(t)\rangle\langle\psi_n(t)|$$

is the density matrix in terms of adiabatic approximation. This way of definition is in fact a natural extension of the ordinary Berry's phase. For an isolated system, it is clear that γ is an anti-symmetric matrix, and $\gamma_{mn}(t) = \zeta_m(t) - \zeta_n(t)$. Obviously, γ contains less information than the traditional Berry's phases $\zeta_m(t)$ do. Besides, it is also interesting to notice that either pure state or mixed state leads to the same Berry's phase matrix (see Sjöqvist *et al.*, 2000, a recent work on defining the geometric phase of mixed states in detail). Obviously, this method of investigating the dissipation geometric phase differs from those of (Garrison and Wright, 1988; Gamliel and Freed, 1989).

3. LEGGETT'S DISSIPATION MODEL

Let us consider a spin-1/2 particle in an external magnetic field B_0 pointing in the z-direction, as a physical realization of qubit. Two eigenstates of the spin operator σ_z, separated by an energy gap $\omega_0 = gB_0$ (g is the gyromagnetic ratio), are labeled as the logic basis $|0\rangle$ (spin down) and $|1\rangle$ (spin up). It is well known that the Bloch vector of this single qubit rotates along the z-axis with an angular frequency ω_0. We firstly consider the special case without the periodical field, namely, a physical system only composed by a single qubit and its environment. The Hamiltonian reads

$$H = \frac{1}{2}\sigma_z\omega_0 + \sum_k b_k^\dagger b_k\omega_k + \sum_k \sigma_z(g_k b_k^\dagger + g_k^* b_k), \tag{3}$$

which is equivalent to the famous model in connection with the tunneling problem introduced by Leggett *et al.* (see also Unruh, 1995). In this case, one could conveniently move into the interaction picture by applying the time evolution operator:

$$U(t) = \exp\left[-i\int_0^t \sum_k \sigma_z(g_k b_k^\dagger e^{i\omega_k\tau} + g_k^* b_k e^{-i\omega_k\tau})d\tau\right]$$
$$= \exp\left[\sigma_z \sum_k (b_k^\dagger \alpha_k(t) - b_k \alpha_k^*(t))\right], \tag{4}$$

with $\alpha_k(t) = g_k(1 - e^{i\omega_k t})/\omega_k$. Obviously $U(t)$ has the form of conditional displacement operator for the field, with the sign decided by the logical value of the qubit. It could be easily seen that $U(t)$ has the role to realize the entanglement of qubit states and field states, for instance, an initial superposition state of the qubit and a vacuum state for the field mode would be transformed into the following entangled state

$$U(t):\ (c_0|0\rangle + c_1|1\rangle) \otimes |0_k\rangle$$
$$\mapsto c_0|0\rangle|-\alpha_k(t)\rangle + c_1|1\rangle|+\alpha_k(t)\rangle, \tag{5}$$

where $|\alpha_k(t)\rangle$ is a Glauber coherent state. It is clear that this entanglement could lead to decoherence of the qubit state. The docoherence factor could easily be derived by following the method shown in Fujikawa and Ono (1996) and Sun *et al.* (1998), which we will not explore here.

Now we would like to consider a simple dissipation model with a time-dependent magnetic field along the z-axis, which is just the non-static version of the above Hamiltonian by replacing ω_0 with $\omega(t) = \mu B(t)$. It will be seen that in this simple model, the dissipation environment has an impact on the geometric phase gained by the qubit because of the existence of the periodical field. The generalization to the more realistic situation with an oscillating field in the xy-plane (Jones *et al.*, 1998) will be investigated elsewhere.

The state of our model is described by a density operator $\varrho(t)$ which at time $t = 0$ is of the form $\varrho(0) = \rho(0) \otimes \prod_k R_{kT}$, where R_{kT} is the thermal density matrix of the k-th mode of the field. Taking into account of the observation $[\sigma_z, H] = 0$, $\rho_{ii}(t) = \rho_{ii}(0)$, $(i = 0, 1)$, and using the unitary transformation (g_k being real number)

$$V = \exp\left[\sum_k \frac{g_k}{\omega_k}(b_k^\dagger - b_k)\right], \tag{6}$$

we could make the Hamiltonian decoupled and the density matrix elements is obtained as

$$\rho_{ij}(t) = \langle i|Tr_R U(t)\varrho(0)U^{-1}(t)|j\rangle. \tag{7}$$

The coherence ρ_{10} becomes

$$\rho_{10}(t) = \exp[-i\Omega(t)]\rho_{10}(0)\prod_k Tr_k M_{kT}, \tag{8}$$

where $\Omega(t) = \int_0^t \omega(\tau)d\tau$,

$$\begin{aligned} M_{kT} &= V^{-1}(1)\exp(-it\omega_k b_k^\dagger b_k)V(1)R_{kT} \\ &\quad \times V^{-1}(-1)\exp(it\omega_k b_k^\dagger b_k)V(-1), \end{aligned} \tag{9}$$

and $V(\pm 1) = \exp[\pm 1 \sum_k (g_k/\omega_k)(b_k^\dagger - b_k)]$. Careful calculations show that

$$\rho_{10}(t) = \exp[-i\omega(t)t]\rho_{10}(0)\exp[-\tilde{\Gamma}(t, T)],$$

$$\tilde{\Gamma}(t, T) = \sum_k \left[4(g_k/\omega_k)^2(1 - \cos\omega_k t)\coth\frac{\omega_k}{2T}\right], \tag{10}$$

which, in the continuum limit, could be written as

$$\begin{aligned} \tilde{\Gamma}(t, T) &= \int d\omega \frac{dk}{d\omega} G(\omega) 4[g(\omega)/\omega]^2 \\ &\quad \times (1 - \cos\omega t)(1 + 2\langle n(\omega)\rangle_T), \end{aligned} \tag{11}$$

where $G(\omega)$ is the density of modes at frequency ω (the index k is dropped), $\langle n(\omega)\rangle_T$ is just the Bose–Einstein distribution for the thermal photons at temperature T, and $(dk/d\omega)$ is the dispersion relation. By applying the 3D Ohmic spectral density function (Leggett *et al.*, 1987; Vacchini, 2000), we could find the final result with a rather complex form, which we will not outline here since it is not very important for the present purpose. The key point is, the Berry's phase matrix reads

$$(\tilde{\gamma}(t, T))_{10} = i\tilde{\Gamma}(t, T), \tag{12}$$

which represents a decaying term due to the environmental noise. Note that this method is different from two early works which used the master equation under the Born–Markov approximation (Gamliel and Freed, 1989) or the non-hermitian Hamiltonian method (Garrison and Wright, 1988). Besides, we would like to point out that our main results derived in the framework of deduced density matrix keep valid for both 1D and 3D cases: they differ from each other only in the concrete form of $\tilde{\Gamma}(t, T)$ which depends on the explicit forms of 1D and 3D Ohmic functions.

As an example, here we consider the 1D Ohmic function $G(\omega) = (\eta\omega e^{-\omega/\omega_c})/g^2(\omega)$, where ω_c is some cut-off frequency. The final result about

$\tilde{\Gamma}(t, T)$ is

$$\tilde{\Gamma}(t, T) = \frac{\eta}{2} \ln\left(1 + \omega_c^2 t^2\right) + \eta \sum_{k=1}^{\infty} \ln\left[1 + \left(\frac{\omega_c T t}{T + k\omega_c}\right)^2\right],$$

which, in the low temperature limit ($\omega_c \gg T$), reduces to

$$\tilde{\Gamma}(t, T) = \frac{\eta}{2} \ln\left(1 + \omega_c^2 t^2\right) + \eta \ln\left[\frac{1}{\pi T t} \sinh(\pi T t)\right] \tag{13}$$

which evolves from the square law $\tilde{\Gamma}(t, T) \sim \omega_c^2 t^2$ in short time limits to the linear law $\tilde{\Gamma}(t, T) \sim Tt$ in long time limits. Here one should note that the first term in the above equation is independent of the temperature, which in fact represents the part from the inevitable vacuum fluctuation.

4. GEOMETRIC PHASE IN TWO-LEVEL-SYSTEM RESERVOIR

Our concerned system is still a spin-1/2 particle trapped in periodical magnetic field $B(t)$ along the z-axis. For weak couplings, every excitation of the heat bath can be regarded as a quantum transition occurring in an individual two-level (sub)system, which means that we can also resort to a thermal reservoir composed of an infinite number of two-level systems as an equally universal environment in physical realistic viewpoints to probe the dissipation effects in our simple model system (Caldeira *et al.*, 1993; Fujikawa and Ono, 1996; Shao *et al.*, 1996; Shao and Hänggi, 1998; Sun *et al.*, 1998; Mozyrsky and Privman, 2000).

Since the original model in (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000) has rather complex quantum behaviors and in particular, it is generally not exactly solvable, here, to give an intuitive understanding of this new model, we still consider a relatively simple Hamiltionian with a non-demolition coupling (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Fortunato *et al.*, 1999; Mozyrsky and Privman, 2000), namely

$$\mathcal{H}(t) = \frac{1}{2}\sigma_z \omega(t) + \sum_s \omega_s \sigma_{zs} + \sigma_z \sum_s g_s \sigma_{xs}, \tag{14}$$

where $\omega(t) = \mu B(t)$. The state of our model system is described by a density operator $\varrho(t)$ which at time $t = 0$ is of the form $\varrho(0) = \rho(0) \otimes \prod_s R_{sT}$, where R_{sT} is the thermal density matrix of s-th mode of the field. It is easy to observe that $[\sigma_z, H] = 0$ and $\rho_{ii}(t) = \rho_{ii}(0)$, $(i = 0, 1)$, which are in fact a main feature of the non-demolition coupling model.

Taking into account of the well-known formula $\varrho(t) = \exp(-i\mathcal{H}t)\varrho(0)\exp(i\mathcal{H}t)$ and the denoted expression of density matrix elements of the whole model system $\rho_{ij}(t) \equiv \langle i|Tr_R\varrho(t)|j\rangle$, one could write the coherence ρ_{10} as

$$\rho_{10}(t) = \exp[-i\Omega(t)]\rho_{10}(0)\prod_s Tr_s M_{sT}, \tag{15}$$

where $\Omega(t) = \int_0^t \omega(\tau)d\tau$, $M_{sT} = V_s^\dagger(1)R_{sT}V_s(-1)$, and $V_s(\pm 1) = \exp[i(\omega_s\sigma_{z_s} \pm g_s\sigma_{x_s})t]$. Through a straightforward calculation, it is easy to obtain a result analogous to that obtained within the standard Leggett's model as

$$\rho_{10}(t) = \exp[-i\Omega(t)]\rho_{10}(0)\exp(-\Gamma(t)),$$

$$\Gamma(t) = \sum_s \ln\,[1 - 2(g_s/\varpi_s)^2\sin^2(\varpi_s t)]^{-1}, \tag{16}$$

where $\varpi_s = \sqrt{\omega_s^2 + g_s^2}$, and we have used the following identity

$$\exp[i(\lambda_1\sigma_z + \lambda_2\sigma_x)] = \cos\lambda + i\frac{\sin\lambda}{\lambda}(\lambda_1\sigma_z + \lambda_2\sigma_x),$$

with $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}$. In the continuum limits, the decaying term also could be rewritten as

$$\Gamma(t) = \int d\omega\frac{ds}{d\omega}G(\omega)\ln\,[1 - 2(g_s/\varpi_s)^2\sin^2(\varpi_s t)]^{-1}, \tag{17}$$

where $G(\omega)$ is the density of modes at frequency ω (the index s is dropped) and $(ds/d\omega)$ is the dispersion relation. Here, it is very interesting to observe that the obtained decaying term $\Gamma(t)$ is independent of temperature, which coincides with the important conclusions derived for weak coupling in (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000), i.e., the quantum coherence or the oscillatory decay of coherence sustains up to infinite temperature (see also Fujikawa and Ono, 1996; Sun *et al.*, 1998). It is evident that this decaying term is quite different from that obtained by using the traditional Leggett's dissipation model, since the later depends both the evolution time and the temperature. This difference can be traced back to the severe restriction restriction of the thermal induced excitation possibilities of the reservoir degrees of freedom (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000). In particular, the Berry's phase matrix now reads

$$(\gamma(t))_{10} = i\Gamma(t), \tag{18}$$

which, as that in previous case, also represents a damping process due to the environmental noise. For the purpose of comparison, we still consider the 1D Ohmic function given above, thus the final result about $\Gamma(t)$ in weak coupling limits ($g_s \to 0$) could be written in a very simple form as

$$\Gamma(t) = \frac{\eta}{2} \ln(1 + 4\omega_c^2 t^2), \tag{19}$$

which, in the experimentally accessible domain of time ($\omega_c t \gg 1$), reduces to $\Gamma(t) = \eta \ln(2\omega_c t)$. In comparison with previous result in low temperature limits ($\omega_c \gg T$), one could see that, the later would give a linear law with respect to both evolution time and temperature $\tilde{\Gamma}(t, T) \sim Tt$ in long time limits; but in short time limits, there is a square law only with respect to the evolution time $\Gamma(t) \sim \omega_c^2 t^2$ for both of two models. Evidently, at zero temperature, as pointed out in (Caldeira *et al.*, 1993; Shao *et al.*, 1996; Shao and Hänggi, 1998; Mozyrsky and Privman, 2000), the dissipation dynamics of the two models is essentially identical.

5. CONCLUSION AND REMARKS

In conclusion, we have studied the impact of environmental dissipation on the geometric phase acquired by a spin trapped in a periodical magnetic field, based on two dissipation models respectively. In order to grasp the main physical feature of this model system, we resort to an exactly solvable non-demolition coupling and the results show that, for both of two models, the environmental noise will lead to a decaying term in the matrix of Berry's phase, which corresponds to a decoherence process of the spin (qubit). However, the forms of the geometric phase are quite different for two dissipation models, in particular, the geometric phase is independent on the temperature for the new type of two-level-system reservoir. This observation not only clearly shows its difference from that based on traditional Leggett's model, but also shed some new lights on the decoherence problem particularly in geometric quantum computing efforts (Ekert *et al.*, 2000; Jones *et al.*, 2000). For short time limits or at zero temperature, both of two models exhibit essentially the same behaviors, as it should be. In the same way, one could also study the coupling system of a register of L length with its reservoir if the interaction between spins (qubits) could be ignored (Palma *et al.*, 1996). Of course, our investigation here is just the beginning point to understand the effects of dissipation on Berry's phase, in particular, our model did not consider the more complex type of dissipation with the energy transfer between spin and reservoir (Caldeira *et al.*, 1993; Fujikawa and Ono, 1996; Shao *et al.*, 1996; Shao and Hänggi, 1998; Sun *et al.*, 1998; Mozyrsky and Privman, 2000), which may comprise the challenge for further works in the future.

ACKNOWLEDGMENTS

H. J. is grateful to Prof. Mo-Lin Ge for helpful discussions. This research was supported in part by the National Science Foundation (NSF) of China under No.10304020 and the Wuhan Youth Chenguang project.

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